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Marginal extended perturbations in two dimensions and gap–exponent relations

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Abstract. The most general form of a marginal extended perturbation in a two-dimensional system is deduced from scaling considerations. It includes as particular cases extended perturbations decaying either from a surface, a line or a point for which exact results have been previously obtained. The first-order corrections to the local exponents, which are functions of the amplitude of the defect, are deduced from a perturbation expansion of the two-point correlation functions. Assuming covariance under conformal transformation, the perturbed system is mapped onto a cylinder. Working in the Hamiltonian limit, the first-order corrections to the lowest gaps are calculated for the Ising model. The results confirm the validity of the gap–exponent relations for the perturbed system.

1. Introduction

Following their application to semi-infinite two-dimensional systems [1], the methods of conformal invariance have been used to study the local critical behaviour near line defects [2–4] and star-shaped defects [5] in the two-dimensional Ising model. Although the symmetries usually considered to be necessary for conformal invariance [6, 7] (dilatation, rotation and translation invariance) are partially broken by such perturbations, the properties associated with conformal invariance are preserved and the spectrum-generating algebra has even been identified [8–10].

There are some indications that scaling operators remain covariant under conformal transformation in the presence of marginal extended perturbations too. This type of defect, which was originally introduced as a deviation from the bulk coupling strength with a power-law decay from a free surface in the 2D Ising model [11], has been extensively studied in the last years [12–17]. It becomes marginal when the decay exponent is equal to the scaling dimension of the coupling (see section 2). Then local exponents may vary with the perturbation amplitude. When the system is mapped onto a strip, provided the perturbation profile is properly transformed, the gap–exponent relations and the tower-like structure of the spectrum are preserved [18–20] but the associated algebra has not yet been determined. Similar results were obtained with an extended perturbation induced by an internal line defect [21]. This type of perturbation was first considered in [22] (see also [23–25]). In the case of a radial extended perturbation [26, 27] or with a decaying surface field [28], the gap–exponent relations are also satisfied although the equidistant-level structure of the spectrum is lost.

In the present work we consider the most general marginal extended perturbation in the 2D Ising model and show that, up to first order in the defect amplitude, the gap–exponents relations remain valid. The form of the perturbation is obtained through scaling considerations in section 2. The first-order corrections to the local exponents are deduced from a perturbation calculation of the two-point correlation functions to logarithmic accuracy in section 3. This is followed by a determination of the transformed profile in the strip geometry and a study of the condition for shape invariance under the special conformal transformation in section 4. The first-order shifts of the lowest gaps are calculated and the validity of the gap–exponent relations examined in section 5. Specific examples are discussed in section 6.

2. Scaling considerations

In a continuum description let the Hamiltonian of a two-dimensional inhomogeneous system be written as

$$-\beta H = -\beta H_c + \int \Delta(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}^2 r \tag{1}$$

where H_c is the critical point Hamiltonian of a conformally invariant system, $\Delta(\boldsymbol{r})$ is an extended perturbation, with scaling dimension y_ϕ, which is conjugate to the local operator $\phi(\boldsymbol{r})$ with bulk scaling dimension $x_\phi = 2 - y_\phi$. This perturbation can be written as the product of an amplitude g by a shape function $\mathcal{Z}(\boldsymbol{r})$ which gives the form of the inhomogeneity. It is assumed to be covariant under rescaling so that, in polar coordinates,

$$\mathcal{Z}\left(\frac{r}{b}, \theta\right) = b^\omega \mathcal{Z}(r, \theta) \tag{2}$$

where the scale invariance of the angle θ has been taken into account. With $b = r$ equation (2) leads to a power-law behaviour for the radial part

$$\mathcal{Z}(r, \theta) = f(\theta)/r^\omega \tag{3}$$

whereas the angular dependence remains arbitrary.

The inhomogeneity $\Delta(\boldsymbol{r})$ transforms according to

$$\Delta'(\boldsymbol{r}') = g'\mathcal{Z}\left(\frac{r}{b}, \theta\right) = b^{y_\phi}\Delta(\boldsymbol{r}) = b^{y_\phi - \omega} g\mathcal{Z}\left(\frac{r}{b}, \theta\right). \tag{4}$$

As a consequence, the perturbation amplitude scales as follows [29, 30]

$$g' = b^{y_\phi - \omega} g. \tag{5}$$

When $\omega > y_\phi$ the perturbation decays strongly enough for the extended perturbation to be irrelevant. When $\omega < y_\phi$, on the contrary, the amplitude increases under rescaling, the perturbation is relevant and the original fixed point is unstable. More interesting is the borderline situation where $\omega = y_\phi$. Then the extended perturbation is marginal and non-universal (g-dependent) local exponents are expected. These results are easily generalized in higher dimensions.

3. Perturbation theory

We now specialize to the case where the perturbation term in (1) is marginal and involves the energy density operator $\varepsilon(\boldsymbol{r})$. Then

$$-\beta H = -\beta H_c + g\int \mathcal{Z}(\boldsymbol{r})\varepsilon(\boldsymbol{r})\,\mathrm{d}^2r. \tag{6}$$

The first-order change in the local critical behaviour can be deduced from a perturbation expansion of the correlation functions [22]. The order parameter two-point correlation function has the following expansion in powers of g

$$G_{\sigma\sigma}(\boldsymbol{R}, g) = \sum_{n=0}^{\infty}\frac{g^n}{n!}\int \langle\langle\sigma(0)\sigma(\boldsymbol{R})\varepsilon(\boldsymbol{r}_1)\varepsilon(\boldsymbol{r}_2)\cdots\varepsilon(\boldsymbol{r}_n)\rangle\rangle\prod_{i=1}^{n}\mathcal{Z}(\boldsymbol{r}_i)\,\mathrm{d}^2r_i \tag{7}$$

where the double brackets denote the irreducible part of a multi-point correlation function and $\boldsymbol{R}$ is far from the defect in the bulk of the system. The nth-order contribution can be rewritten as

$$\delta G_{\sigma\sigma}^{(n)} = g^n\int_{r_1<r_2\cdots<r_n}\langle\langle\sigma(0)\sigma(\boldsymbol{R})\varepsilon(\boldsymbol{r}_1)\varepsilon(\boldsymbol{r}_2)\cdots\varepsilon(\boldsymbol{r}_n)\rangle\rangle\prod_{i=1}^{n}\mathcal{Z}(\boldsymbol{r}_i)\,\mathrm{d}^2r_i \tag{8}$$

where the $n!$ in the prefactor disappears due to the ordering on the r_i. The main contribution to the multiple integral comes from regions where operators are grouped pairwise close together. Then the operator product expansion can be used to reduce the multi-point function. With a perturbation in the bulk of the system the following reduction relations are needed

$$\sigma(\boldsymbol{r}_1)\varepsilon(\boldsymbol{r}_2) \simeq a\sigma(\boldsymbol{r}_1)r_{12}^{-x_\varepsilon} \qquad \varepsilon(\boldsymbol{r}_1)\varepsilon(\boldsymbol{r}_2) \simeq b\varepsilon(\boldsymbol{r}_1)r_{12}^{-x_\varepsilon}. \tag{9}$$

The coefficient b in the second relation vanishes when the system is invariant under duality ($\varepsilon \to -\varepsilon$). Then the expansion should be continued one step further. This occurs for the Ising model only in the bulk [31] since the surface case does not possess the duality symmetry.

Let us first suppose that r_{01} is the smallest distance entering the correlation function in (8). Then the first relation in (9) can be used and the integration over r_1 carried out to give

$$a\sigma(0)\int_1^{r_2}\mathrm{d}r_1 r_1^{1-\omega-x_\varepsilon}\int_0^{2\pi}f(\theta_1)\,\mathrm{d}\theta_1. \tag{10}$$

Since in the marginal case $\omega = 2 - x_\varepsilon$, the first integral contributes a factor $\ln r_2$. It may be checked that when one first contracts other pairs, like $\varepsilon(\boldsymbol{r}_1)\varepsilon(\boldsymbol{r}_2)$, the result is logarithmically smaller. It follows that, to the leading logarithmic order,

$$\delta G_{\sigma\sigma}^{(n)} = g^n a S_f\int_{r_2<r_3\cdots<r_n}\ln r_2\langle\langle\sigma(0)\sigma(\boldsymbol{R})\varepsilon(\boldsymbol{r}_2)\cdots\varepsilon(\boldsymbol{r}_n)\rangle\rangle\prod_{i=2}^{n}\mathcal{Z}(\boldsymbol{r}_i)\,\mathrm{d}^2r_i \tag{11}$$

where S_f is the angular integral in equation (10). The same process can be iterated n times leading to

$$\delta G_{\sigma\sigma}^{(n)} = \frac{1}{n!}[gaS_f\ln R]^n\langle\langle\sigma(0)\sigma(\boldsymbol{R})\rangle\rangle \tag{12}$$

Here the $n!$ is restored through the successive integrations involving increasing powers of a logarithm. Finally the perturbation series can be summed giving

$$G_{\sigma\sigma}(\boldsymbol{R}, g) = G_{\sigma\sigma}(\boldsymbol{R}, 0) R^{gaS_f} \tag{13}$$

to logarithmic accuracy. The order-parameter local scaling dimension can be read out of the correlation function. It is indeed non-universal and reads

$$x_\sigma(g) = x_\sigma - ga \int_0^{2\pi} f(\theta)\, \mathrm{d}\theta + \mathrm{O}(g^2). \tag{14}$$

The same method can be applied step by step to the calculation of the energy density correlation function using the second reduction relation in equation (9). The local scaling dimension is then

$$x_\varepsilon(g) = x_\varepsilon - gb \int_0^{2\pi} f(\theta)\, \mathrm{d}\theta + \mathrm{O}(g^2). \tag{15}$$

Let us now consider the case of an extended perturbation centred on the surface of a semi-infinite system [32]. In the direction perpendicular to the free surface translation invariance is lost and the operator product expansion has to be modified accordingly. When the first point is the origin, located on the surface, the structure constants in (9) only acquire an angular dependence

$$\sigma(0)\varepsilon(\boldsymbol{r}) \simeq a(\theta)\sigma(0) r^{-x_\varepsilon} \qquad \varepsilon(0)\varepsilon(\boldsymbol{r}) \simeq b(\theta)\varepsilon(0) r^{-x_\varepsilon} \tag{16}$$

where θ is the polar angle measured from the surface. When the second point goes to the surface too, i.e. when $\theta \simeq r^{-1}$ or $\pi - r^{-1}$, the operator product decays with an exponent which is the surface scaling dimension $x_\varepsilon^{\mathrm{s}} = d$ [33]. Then one expects quite generally $a(\theta) \sim b(\theta) \sim (\sin\theta)^{2-x_\varepsilon}$.

For the Ising model, with $x_\varepsilon = 1$, one indeed obtains† [32]

$$a(\theta) = \frac{2}{\pi} \sin\theta \qquad b(\theta) = \frac{8}{\pi} \sin\theta \cos^2\theta. \tag{17}$$

The bulk values in (9) are $a = (2\pi)^{-1}$ and $b = 0$ due to the duality symmetry in the 2D bulk Ising system [31]. In this latter case the expansion is in powers of g^2.

The calculation of the two-point correlation functions then proceeds as above in the bulk. The main change is introduced by the angular dependence of the structure constants and one gets the following results for the first-order corrections to the local exponents

$$x_\sigma^{\mathrm{s}}(g) = x_\sigma^{\mathrm{s}} - g \int_0^{\pi} a(\theta) f(\theta)\, \mathrm{d}\theta + \mathrm{O}(g^2) \qquad x_\varepsilon^{\mathrm{s}}(g) = x_\varepsilon^{\mathrm{s}} - g \int_0^{\pi} b(\theta) f(\theta)\, \mathrm{d}\theta + \mathrm{O}(g^2) \tag{18}$$

provided the angular integrals are not singular. The same restriction applies to the bulk results in equations (14) and (15).

† The polar angle was measured from the perpendicular direction in [32].

4. Conformal aspects

In this section we assume that conformal techniques can still be used in the presence of a general extended marginal perturbation. The conformal mapping of the perturbed system, either infinite or semi-infinite, onto a strip will be used in the next section to get the perturbed gaps, allowing a comparison with the perturbation results for the local exponents.

4.1. Plane-to-cylinder transformation

We use the mapping $w = (L/2\pi)\ln z$ of the full plane $z = r\exp(\mathrm{i}\theta)$ onto a cylinder $w = u + \mathrm{i}v$, $-\infty < u < +\infty$, $0 < v < L$ [34]. Under this transformation the marginal perturbation $\Delta(z)$ with scaling dimension y_ϕ is changed into [18]

$$\Delta(w) = b(z)^{y_\phi}\Delta(z) \tag{19}$$

with a dilatation factor given by

$$b(z) = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-1} = \frac{2\pi r}{L}. \tag{20}$$

The shape function transforms in the same way since g is invariant. Using

$$r = \exp 2\pi u/L \qquad \theta = 2\pi v/L \tag{21}$$

one obtains

$$\Delta(u, v) = g(2\pi/L)^{y_\phi} f(2\pi v/L) \tag{22}$$

where translation invariance along the strip, which is directly linked to the marginal behaviour, is essential for the conformal properties. The perturbation is generally inhomogeneous in the transverse direction although translation invariance is preserved in the case of a radial perturbation [26, 27].

4.2. Special conformal transformation

The infinitesimal special conformal transformation also plays an important role. When a system is invariant under this transformation, differential equations for correlation functions or profiles can be obtained, leading to the asymptotic behaviour of the first and completely determining the second [1, 35].

Let the transformation be written as

$$z' = z + \epsilon z^2 \qquad w = z' = r'\exp(\mathrm{i}\theta'). \tag{23}$$

Up to $\mathrm{O}(\epsilon)$, one obtains

$$r' = r + \epsilon r^2\cos\theta \qquad \theta' = \theta + \epsilon r\sin\theta \qquad b(r,\theta) = 1 - 2\epsilon r\cos\theta. \tag{24}$$

Together with (19), this leads to the following transformed perturbation

$$\Delta(r', \theta') = (1 - 2\epsilon r\cos\theta)^{y_\phi} g\frac{f(\theta)}{r^{y_\phi}} = \left[1 - \epsilon r\sin\theta\left(y_\phi\cot\theta + \frac{\mathrm{d}\ln f}{\mathrm{d}\theta}\right)\right] g'\frac{f(\theta')}{r'^{y_\phi}}. \tag{25}$$

The perturbation is shape-invariant when the coefficient of ϵ vanishes, i.e. when

$$\Delta(r,\theta) = g/|r\sin\theta|^{y_\phi}. \tag{26}$$

This corresponds to an extended perturbation decaying either from the surface of a semi-infinite system [11] or from a line defect in the bulk [22–25, 21]. Then the correlation functions satisfy the same differential equation as in the surface case [1]. Their asymptotic behaviour can also be obtained, with defect exponents replacing surface ones. Transforming the correlation functions via the plane-to-cylinder mapping, the gap–exponent relation and the tower-like structure of the spectrum immediately follows, in agreement with exact results for these perturbations [18, 21].

5. Gap–exponent relations

5.1. Hamiltonian limit and rescaling

We now consider a temperature-like extended marginal perturbation in the 2D Ising model at the bulk critical point. On the cylinder the original perturbation is transformed into (22) with $\phi = \varepsilon$ and $y_\varepsilon = 1$. In the case of a semi-infinite original system $(0 < \theta < \pi)$, the transformed system is a strip with free boundary conditions (FBC) at $v = 0$ and $v = L/2$ on the cylinder whereas for an infinite original system, periodic boundary conditions (PBC) have to be taken at $v = L$.

We work on a square lattice in the extreme anisotropic limit, with unperturbed two-spin interactions $K_1 \gg 1$ in the time direction along the cylinder axis and $K_2 \ll 1$ in the transverse one. At the bulk critical point $K_2 = K_1^*$ where K_1^* is a dual coupling satisfying $\tanh(K_1^*) = \exp(-2K_1)$. The perturbation is assumed to act only on the transverse interactions K_2.

In the Hamiltonian limit [36, 7] the system becomes anisotropic with a correlation length ratio [37]

$$\frac{\xi_2}{\xi_1} = \frac{\cosh(2K_2)}{\cosh(2K_1)} \simeq 2K_1^*. \tag{27}$$

Isotropy can be restored by rescaling the lattice parameter a_1 in the time direction to $a_1 = 2K_1^*$ measured in units of a_2 in the transverse direction. The row-to-row transfer operator may be written as

$$\mathcal{T} = \exp(-a_1\mathcal{H}). \tag{28}$$

The Ising Hamiltonian then takes the form

$$\mathcal{H} = -\frac{1}{2}\left[\sum_n \sigma_n^z + \sum_n \sigma_n^x\sigma_{n+1}^x\right] - g\frac{2\pi}{L}\sum_n f\left(\frac{2\pi n}{L}\right)\sigma_n^x\sigma_{n+1}^x \tag{29}$$

where the σs are Pauli spin operators. The first part in (29) corresponds to a homogeneous system at the bulk critical point, properly normalized to give critical excitations with velocity $v_s = 1$ [38]. The perturbation term keeps the same amplitude as in the continuum expression (22) since, lengths being measured in units of a_2, $\exp(-\mathcal{H})$ operates a transfer by one unit length in the time direction and each perturbed bond in $\mathcal{H}$ is associated with one surface unit on the isotropic rescaled system.

5.2. Diagonalization of the unperturbed Hamiltonian

The unperturbed part of the Hamiltonian in (29) is rewritten as a quadratic form in fermion operators through a Jordan–Wigner transformation [39]

$$\mathcal{H}_c(P) = -\sum_{n=1}^{N}\left(c_n^+c_n - \frac{1}{2}\right) - \frac{1}{2}\sum_{n=1}^{N-1}(c_n^+ - c_n)(c_{n+1}^+ + c_{n+1}) + \frac{1}{2}P(c_N^+ - c_N)(c_1^+ + c_1) \tag{30}$$

where the chain length N is assumed to be even in the following. For a semi-infinite original system one has to take $N = L/2$ and $P = 0$ in (30). If the original system covers the whole plane then $N = L$ and $P = \pm 1$ is an eigenvalue of the parity operator $\mathcal{P} = \exp(\mathrm{i}\pi \sum_n c_n^+ c_n)$ which commutes with $\mathcal{H}_c$.

The Hamiltonian is put under diagonal form

$$\mathcal{H}_c(P) = \sum_k \varepsilon_k (\eta_k^+ \eta_k - \tfrac{1}{2}) \tag{31}$$

through a canonical transformation [40, 41]. The squares of the excitation energies for the diagonal fermions in (31), $\varepsilon_k = 2|\sin k/2|$, can be obtained as the solutions of two equivalent eigenvalue problems with normalized eigenvectors, Φ_k and Ψ_k. The wavevector quantization depends on the boundary conditions, i.e. on P:

$$k(P=+1) = (2p+1)\frac{\pi}{N} \qquad k(P=-1) = 2p\frac{\pi}{N} \qquad -\frac{N}{2} \leqslant p \leqslant \frac{N}{2} - 1 \qquad \text{(PBC)} \tag{32a}$$

$$k(P=0) = (2p+1)\frac{\pi}{2N+1} \qquad 0 \leqslant p \leqslant N-1 \qquad \text{(FBC)}. \tag{32b}$$

With PBCs, the normalized eigenvectors are given by

$$\Phi_k(n) = (-1)^n \sqrt{\frac{2}{N}} \sin kn \qquad \Psi_k(n) = (-1)^n \sqrt{\frac{2}{N}} \cos k\left(n + \frac{1}{2}\right) \qquad 0 < k < \pi \tag{33a}$$

$$\Phi_k(n) = (-1)^n \sqrt{\frac{2}{N}} \cos kn \qquad \Psi_k(n) = (-1)^n \sqrt{\frac{2}{N}} \sin k\left(n + \frac{1}{2}\right) \qquad -\pi < k < 0 \tag{33b}$$

$$\Phi_0(n) = -\Psi_0(n) = \frac{(-1)^n}{\sqrt{N}} \qquad \Phi_{-\pi}(n) = -\Psi_{-\pi}(n) = \frac{1}{\sqrt{N}} \qquad (P=-1) \tag{33c}$$

whereas one obtains

$$\Phi_k(n) = (-1)^n \frac{2}{\sqrt{2N+1}} \cos k\left(n - \frac{1}{2}\right) \qquad \Psi_k(n) = (-1)^{n+1} \frac{2}{\sqrt{2N+1}} \sin kn \tag{34}$$

with FBCs.

5.3. Perturbation theory

Up to first order in the perturbation amplitude, the levels of $\mathcal{H}_c$ in (29) are shifted by

$$E_\alpha(g) - E_\alpha(0) = -g\frac{2\pi}{L} \sum_n f\left(\frac{2\pi n}{L}\right) \langle\alpha|\sigma_n^x \sigma_{n+1}^x|\alpha\rangle. \tag{35}$$

The relevant levels in the following are the ground state $|0\rangle$ and the states $|\sigma\rangle$ and $|\varepsilon\rangle$ of the unperturbed Hamiltonian which are involved in the calculation of the lowest gaps. These states are the lowest ones with non-vanishing matrix elements $\langle 0|\sigma_n^x|\sigma\rangle$ and $\langle 0|\sigma_n^x \sigma_{n+1}^x|\varepsilon\rangle$ with the ground state.

With periodic boundary conditions and N even, $|0\rangle$ is the vacuum of $\mathcal{H}_c(+1)$. $|\varepsilon\rangle = \eta_0^+\eta_1^+|0\rangle$, which is even, also belongs to the spectrum of $\mathcal{H}_c(+1)$ and contains the two lowest excitations corresponding to $p = 0$ and $p = -1$ in $k(P = +1)$. $|\sigma\rangle = \eta_0^+|0\rangle_{(-1)}$, which is odd, belongs to the spectrum of $\mathcal{H}_c(-1)$. It contains a single excitation with vanishing wavevector corresponding to $p = 0$ in $k(P = -1)$. As a consequence this state is degenerate with the vacuum $|0\rangle_{(-1)}$ of $\mathcal{H}_c(-1)$.

In the case of FBCs, the ground state $|0\rangle$ is the vacuum of $\mathcal{H}_c(0)$, $|\varepsilon\rangle$ is defined as above with two excitations corresponding to $p = 0$ and $p = 1$ in $k(P = 0)$ while $|\sigma\rangle = \eta_0^+|0\rangle$ only contains the lowest one.

The matrix elements in (35), like the Hamiltonian, are obtained making use of the Jordan–Wigner and canonical transformations and read

$$D_0(n) = \langle 0|\sigma_n^x\sigma_{n+1}^x|0\rangle = -\sum_k \Psi_k(n)\Phi_k(n+1) \tag{36a}$$

$$D_\sigma(n) = \langle \sigma|\sigma_n^x\sigma_{n+1}^x|\sigma\rangle = D_0(n) + 2\Psi_0(n)\Phi_0(n+1) \tag{36b}$$

$$D_\varepsilon(n) = \langle \varepsilon|\sigma_n^x\sigma_{n+1}^x|\varepsilon\rangle = D_0(n) + 2[\Psi_0(n)\Phi_0(n+1) + \Psi_1(n)\Phi_1(n+1)]. \tag{36c}$$

In the case of PBCs, the final expressions for the boundary terms ($n = N$, $n + 1 = 1$) have to be multiplied by $-P$ as in (30) and the eigenvectors Φ_k and Ψ_k are those involved in the diagonalization of $\mathcal{H}_c(P)$ where P is the parity of the states in the matrix elements. Then, using (32)–(34), one obtains

$$D_0(n) = \left(N\sin\frac{\pi}{2N}\right)^{-1} \qquad (P = +1) \tag{37a}$$

$$D_\sigma(n) = \frac{1}{N}\left(1 + \cot\frac{\pi}{2N}\right) \qquad (P = -1) \tag{37b}$$

$$D_\varepsilon(n) - D_0(n) = -\frac{4}{N}\sin\frac{\pi}{2N} \qquad (P = +1) \tag{37c}$$

while for FBCs

$$D_\sigma(n) - D_0(n) = \frac{2}{N}\sin\left(\frac{n\pi}{N}\right) + \mathrm{O}(N^{-2}) \tag{38a}$$

$$D_\varepsilon(n) - D_0(n) = \frac{8}{N}\sin\left(\frac{n\pi}{N}\right)\cos^2\left(\frac{n\pi}{N}\right) + \mathrm{O}(N^{-2}). \tag{38b}$$

According to the gap–exponent relation [34], one expects the scaling dimensions of the operator $\alpha = \sigma, \varepsilon$ to be given by

$$x_\alpha(g) = \lim_{N\to\infty}\frac{N}{2\pi}[E_\alpha(g) - E_0(g)] \qquad x_\alpha^{\mathrm{s}}(g) = \lim_{N\to\infty}\frac{N}{\pi}\left[E_\alpha(g) - E_0(g)\right] \tag{39}$$

which, together with (35) and (36), leads to the first-order changes

$$x_\alpha(g) - x_\alpha(0) = \lim_{N\to\infty} -g\sum_n f\left(\frac{2\pi n}{N}\right)[D_\alpha(n) - D_0(n)] \qquad (\text{PBC}) \tag{40a}$$

$$x_\alpha^{\mathrm{s}}(g) - x_\alpha^{\mathrm{s}}(0) = \lim_{N\to\infty} -g\sum_n f\left(\frac{\pi n}{N}\right)[D_\alpha(n) - D_0(n)] \qquad (\text{FBC}). \tag{40b}$$

In the continuum limit, using (37)–(38) for large N, this transforms into

$$x_\sigma(g) = \frac{1}{8} - \frac{g}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, f(\theta) + \mathrm{O}(g^2) \tag{41a}$$

$$x_\varepsilon(g) = 1 + O(g^2) \tag{41b}$$

for the perturbed scaling dimensions near the source of the inhomogeneity in the plane and

$$x_\sigma^{\mathrm{s}}(g) = \frac{1}{2} - \frac{2g}{\pi}\int_0^{\pi} \mathrm{d}\theta\, f(\theta)\sin\theta + \mathrm{O}(g^2) \tag{42a}$$

$$x_\varepsilon^{\mathrm{s}}(g) = 2 - \frac{8g}{\pi}\int_0^{\pi} \mathrm{d}\theta\, f(\theta)\sin\theta\cos^2\theta + O(g^2) \tag{42b}$$

in the half-plane, in full agreement with the results of section 3.

6. Elliptic defects

As an illustration of the the perturbation results one may consider an extended defect with elliptic symmetry in the 2D Ising model. The angular dependence

$$f(\theta) = (\sin^2\theta + \kappa\cos^2\theta)^{-1/2} \qquad 0 < \kappa < 1 \tag{43}$$

interpolates between the line and radial defects corresponding to $\kappa = 0$ and $\kappa = 1$, respectively.

In the infinite Ising system, using (41a)

$$x_\sigma(g) = \frac{1}{8} - \frac{2g}{\pi}\boldsymbol{K}(\sqrt{1-\kappa}) + \mathrm{O}(g^2) \tag{44}$$

where $\boldsymbol{K}$ is the complete elliptic integral. The radial defect result [26, 27, 42], $x_\sigma(g) = \frac{1}{8} - g + \mathrm{O}(g^2)$ is recovered in the limit $\kappa \to 1$. The correction term displays a logarithmic divergence in the line defect limit, $\kappa \to 0$, which is linked to the singular behaviour of the perturbation at $\theta = 0$ and π. Introducing a cut-off, one gets a jump of the magnetic exponent at $g = 0$ and local order at the bulk critical point for $g > 0$ [21, 31, 43].

In a semi-infinite system (43) corresponds to a defect with its main axis along the surface. Then, using (42)

$$x_\sigma^{\mathrm{s}}(g) = \frac{1}{2} - \frac{4g}{\pi}\frac{\sin^{-1}\sqrt{1-\kappa}}{\sqrt{1-\kappa}} + \mathrm{O}(g^2) \tag{45a}$$

$$x_\varepsilon^{\mathrm{s}}(g) = 2 - \frac{8g}{\pi}\left[\frac{\sin^{-1}\sqrt{1-\kappa}}{(1-\kappa)^{3/2}} - \frac{\sqrt{\kappa}}{1-\kappa}\right] + \mathrm{O}(g^2). \tag{45b}$$

In the surface defect limit, $\kappa = 0$, $x_\sigma^{\mathrm{s}}(g) = \frac{1}{2} - 2g + \mathrm{O}(g^2)$, $x_\varepsilon^{\mathrm{s}}(g) = 2 - 4g + \mathrm{O}(g^2)$ while for the radial defect, $\kappa \to 1$, $x_\sigma^{\mathrm{s}}(g) = 1/2 - 4g/\pi + \mathrm{O}(g^2)$, $x_\varepsilon^{\mathrm{s}}(g) = 2 - 16g/(3\pi) + \mathrm{O}(g^2)$ in agreement with know exact results [18, 26].

When the main axis of the defect is perpendicular to the surface, i.e. with

$$f(\theta) = (\cos^2\theta + \kappa \sin^2\theta)^{-1/2} \qquad 0 < \kappa < 1 \tag{46}$$

one obtains

$$x_\sigma^s(g) = \frac{1}{2} - \frac{4g}{\pi\sqrt{1-\kappa}} \ln\left(\frac{\sqrt{1-\kappa}+1}{\sqrt{\kappa}}\right) + O(g^2) \tag{47a}$$

$$x_\varepsilon^s(g) = 2 - \frac{8g}{\pi}\left[\frac{1}{1-\kappa} - \frac{\kappa}{(1-\kappa)^{3/2}} \ln\left(\frac{\sqrt{1-\kappa}+1}{\sqrt{\kappa}}\right)\right] + O(g^2) \tag{47b}$$

with the same limits as in (45) for the radial defect ($\kappa = 1$). When $\kappa = 0$, i.e. with a line defect perpendicular to the surface, $x_\varepsilon^s(g) = 2 - 8g/\pi + O(g^2)$ whereas the correction to x_σ^s diverges logarithmically. Then one expects, as for a bulk extended line defect, a jump of the surface magnetic exponent at $g = 0$ and local order for enhanced couplings. This could be checked using the techniques of [18, 21].

7. Conclusion

Our main result is the extension of the gap–exponent relations to the case of a general marginal extended perturbation in the 2D Ising model. Although this result was obtained only up to first order in the defect amplitude, one may conjecture that it remains true to all orders, like in the exactly solved limiting cases.

The shape invariance of the perturbation under the special conformal transformation, when it decays like a power of the distance to a line, leads on the cylinder to a spectrum containing conformal towers. At least in this case further work should allow the determination of the spectrum-generating algebra.

When the perturbation expansion for the exponents contains divergent terms, one expects local order at the bulk critical point and a first-order defect transition when the couplings are enhanced as for an internal line defect.

Finally let us mention that our results could be extended to other two-dimensional conformal systems using conformal perturbation theory [6, 7].

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